$f : A \to \mathbb{R},\ x_0 \in A.$

Th3 (Continuity is a local property). Let $f, g : A \to \mathbb{R}$ & $x_0 \in A$ such that $f = g$ on $A \cap V_{\delta_0}(x_0)$ for some $\delta_0 > 0$. Then f is cts at x_0 iff g is cts at x_0.

Th4. (Local Boundness & order preserving) Suppose f is cts at x_0. Then $\exists\ \delta > 0$ & $M > 0$ such that $|f(x)| \le M\ \forall\ x \in V_\delta(x_0) \cap A$,

Moreover if $\alpha < f(x_0) < \beta$ (& f cts at x_0) then $\exists$ $\delta > 0$ such that $\alpha < f(x) < \beta\ \ \forall\ x \in V_\delta(x_0) \cap A$.

Proof. 1st assertion is for you to prove. For the 2nd, let $\varepsilon = \min\{\beta - f(x_0), f(x_0) - \alpha\}$ (>0). Then $\exists\ \delta > 0$ s.t.

$$\underbrace{f(x_0) - \varepsilon}_{\substack{\text{VI} \\ f(x_0) - (f(x_0) - \alpha)}} < f(x) < \underbrace{f(x_0) + \varepsilon}_{\substack{\wedge\text{I} \\ f(x_0) + (\beta - f(x_0))}} \qquad \forall\ x \in V_\delta(x_0) \cap A$$

(Bounded away from zero)

Th5. Let f be cts at x_0 and $f(x_0) \neq 0$. Then $\exists\ \delta > 0$ s.t. $\frac{1}{2}|f(x_0)| \le |f(x)| \le \frac{3}{2}|f(x_0)|\ \forall x \in V_\delta(x_0) \cap A$.

Pf. Let $\varepsilon := \frac{1}{2}|f(x_0)|$ (>0). Then $\exists\ \delta > 0$ s.t.

$$|f(x) - f(x_0)| < \frac{|f(x_0)|}{2} \quad \forall x \in V_\delta(x_0) \cap A$$

$$\text{VI}$$

$$\left\{\begin{array}{l} |f(x)| - |f(x_0)| \\ |f(x_0)| - |f(x)| \end{array}\right.$$

Thm (Chain-Rule) $$\begin{array}{ccccc} T & \xrightarrow{g} & A & \xrightarrow{f} & \mathbb{R} \\ \rotatebox[origin=c]{90}{$\in$} & & \rotatebox[origin=c]{90}{$\in$} & & \\ t_0 & & x_0 = g(t_0) & & \end{array}$$

g cts at t_0 + f cts at $x_0 := g(t_0)$
$\Rightarrow$ $f \circ g$ cts at t_0

Pf. $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $f\left(A \cap V_\delta(x_0)\right) \subseteq V_\varepsilon\left(f(x_0)\right)$

For this $\delta > 0$, $\exists \gamma > 0$ s.t.

$$g\left(T \cap V_\gamma(t_0)\right) \subseteq A \cap V_\delta\left(g(t_0)\right) = V_\delta(x_0) \cap A$$

so $$(f \circ g)\left(T \cap V_\gamma(t_0)\right) \subseteq V_\varepsilon\left(f(x_0)\right) = V_\varepsilon\left((f \circ g)(t_0)\right)$$

re-write: $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$x \in A$, $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

For this δ, $\exists \gamma > 0$ st.

$t \in T$, $|t - t_0| < \gamma \Rightarrow |g(t) - \overset{x_0}{\overset{\|}{g(t_0)}}| < \delta$

Combining, $\forall t \in T$ with $|t - t_0| < \gamma$, one has $g(t) \in A$ & $|g(t) - x_0| < \delta$ so

$$\left|f\left(g(t)\right) - \underset{\underset{f(g(t_0))}{\|}}{f(x_0)}\right| < \varepsilon$$

i.e. $\left|(f \circ g)(t) - (f \circ g)(t_0)\right| < \varepsilon$, valid for all $t \in T$ with $|t - t_0| < \gamma$.